So we have two classes: explicit methods, and implicit methods.
In all cases we want the sequence $\{y_0, y_1, \cdots, y_N\}$ to converge to the sequence $\{y_0, y(t_1), \cdots, y(T)\}$.

If, given a method, we can prove that

$$\exists C > 0 \text{ such that} \qquad \max_n |y_n - y(t_n)| \leq C\Delta t^p$$

with C independent of Δt and $p > 0$, then we say that *the method is convergent*, and *the order of convergence is p* (the bigger p, the faster the convergence).

Theorem 1 (Lax)

If a scheme is **consistent** *and* **stable***, then it is convergent, and the order of convergence is the order of consistency.*

We have however yet to define what *consistency* and *stability* are....

Intuitive explanation of *consistency* and *stability*

- the *consistency error* at a given time step measures the error which is created at that step; it is defined as *the error made when the scheme is applied to the exact solution of the problem*. It can be seen that the consistency error measures of how much the discrete scheme resembles the differential problem.

- *stability* measures how the error, created and accumulated during the previous steps, goes to the next step (is it amplified? does it decay?...)

Consistency of a scheme

Consistency is a measure of how much the discrete scheme resembles the differential problem: the *consistency error* is *the error made when the scheme is applied to the exact solution of the problem*.

The *consistency error* of a given scheme is defined as

$$\tau = \max_{n=1,\dots,N} |\tau_n|$$

where τ_n is the *local truncation error* at the step n, that is defined as:

$$\tau_n = \frac{y(t_{n+1}) - \tilde{y}_{n+1}}{\Delta t}$$

where $\tilde{y}_{n+1}$ is obtained applying one step of the method to the exact solution at the previous time instant $y(t_n)$. We say that the scheme is consistent of order p if:

$$\exists C > 0, \text{ independent of } \Delta t, \text{ s. t. } \tau \leq C \Delta t^p$$

We observe that if the numerical scheme is consistent, then $\tau \to 0$ for $\Delta t \to 0$.

Consistency of Explicit Euler

The exact solution fulfils: $y'(t_n) - f(t_n, y(t_n)) = 0, \quad n = 0, 1, 2, ...$
If we apply the Explicit Euler scheme to the exact solution we will have:

$$\tilde{y}_{n+1} = y(t_n) + \Delta t\, f(t_n, y(t_n))$$

Since $y'(t_n) = f(t_n, y(t_n))$, by Taylor expansion centered in t_n, there exists $z \in [t_n, t_{n+1}]$ such that

$$y(t_{n+1}) = y(t_n) + \Delta t\, y'(t_n) + \frac{\Delta t^2}{2} y''(z) = y(t_n) + \Delta t\, f(t_n, y(t_n)) + \frac{\Delta t^2}{2} y''(z)$$

Thus, the local truncation error becomes

$$\tau_n = \frac{y(t_{n+1}) - \tilde{y}_{n+1}}{\Delta t} = \frac{y(t_{n+1}) - y(t_n)}{\Delta t} - f(t_n, y(t_n)) = \frac{\Delta t}{2} y''(z),$$

and the consistency error is then

$$\tau = \max |\tau_n| \leq \frac{\Delta t}{2} \max_{t \in [t_0, T]} |y''(t)| = C \Delta t.$$

Thus, Explicit Euler scheme is consistent with order of consistency 1.

Consistency of the Heun method

We apply the Heun scheme again to the exact solution $\tilde{y}_n = y(t_n)$:

$$\tilde{y}_{n+1} = y(t_n) + \frac{\Delta t}{2}\left(f(t_n, y(t_n)) + f(t_{n+1}, y(t_n) + \Delta t\, f(t_n, y(t_n)))\right)$$

Since $f(t_n, y(t_n)) = y'(t_n)$, by Taylor expansion centered in t_n, there exists $z_1 \in [t_n, t_{n+1}]$ such that

$$\begin{aligned} y(t_{n+1}) &= y(t_n) + (t_{n+1} - t_n)\, y'(t_n) + \frac{\Delta t^2}{2} y''(z_1) \\ &= y(t_n) + \Delta t\, y'(t_n) + \frac{\Delta t^2}{2} y''(z_1) \end{aligned} \tag{1}$$

and, by Taylor expansion centered in t_{n+1}, there exists $z_2 \in [t_n, t_{n+1}]$ such that

$$\begin{aligned} y(t_n) &= y(t_{n+1}) + (t_n - t_{n+1})\, y'(t_{n+1}) + \frac{\Delta t^2}{2} y''(z_2) \\ &= y(t_{n+1}) - \Delta t\, y'(t_{n+1}) + \frac{\Delta t^2}{2} y''(z_2) \end{aligned} \tag{2}$$

The local truncation error, setting $\hat{y}_{n+1} = y(t_n) + \Delta t\, f(t_n, y(t_n))$ and using again that $y'(t_{n+1}) = f(t_{n+1}, y(t_{n+1}))$ is:

$$\begin{aligned}
\tau_n &= \frac{y(t_{n+1}) - \tilde{y}_{n+1}}{\Delta t} \\
&= \frac{\left(y(t_{n+1}) - y(t_n) - \frac{\Delta t}{2} y'(t_n) - \frac{\Delta t}{2} f(t_{n+1}, \hat{y}_{n+1})\right)}{\Delta t} \\
&= \frac{\left(y(t_{n+1}) - y(t_n) - \frac{\Delta t}{2} y'(t_n) - \frac{\Delta t}{2} y'(t_{n+1}) + \frac{\Delta t}{2} f(t_{n+1}, y(t_{n+1})) - \frac{\Delta t}{2} f(t_{n+1}, \hat{y}_{n+1})\right)}{\Delta t} \\
&= \frac{\left(y(t_{n+1}) - y(t_n) - \frac{\Delta t}{2} y'(t_n) - \frac{\Delta t}{2} y'(t_{n+1})\right)}{\Delta t} + \frac{1}{2}(f(t_{n+1}, y(t_{n+1})) - f(t_{n+1}, \hat{y}_{n+1})) \\
&= A + B,
\end{aligned}$$

where

$$A = \frac{\left(y(t_{n+1}) - y(t_n) - \frac{\Delta t}{2} y'(t_n) - \frac{\Delta t}{2} y'(t_{n+1})\right)}{\Delta t}$$

and

$$B = \frac{1}{2}(f(t_{n+1}, y(t_{n+1})) - f(t_{n+1}, \hat{y}_{n+1}))$$

For the first term we have

$$A = \frac{1}{2}\left(\frac{y(t_{n+1}) - y(t_n)}{\Delta t} - y'(t_n)\right) + \frac{1}{2}\left(\frac{y(t_{n+1}) - y(t_n)}{\Delta t} - y'(t_{n+1})\right)$$
$$= \frac{1}{2}(\frac{\Delta t}{2} y''(z_1)) + \frac{1}{2}(-\frac{\Delta t}{2} y''(z_2)) = \frac{1}{4}\Delta t(z_1 - z_2)\frac{y''(z_1) - y''(z_2)}{z_1 - z_2}$$

where we have used the (1) for the first parenthesis and (2) for the second ones. We then apply the Mean Value Theorem to y'', which states that there exists z_3 between z_1 and z_2 such that: $\frac{y''(z_1)-y''(z_2)}{z_1-z_2} = y'''(z_3)$, and get:

$$|A| = |\frac{1}{4}\Delta t(z_1 - z_2)y'''(z_3)| \leq \frac{\Delta t^2}{4}|y'''(z_3)|$$

For the second term we have

$$
\begin{aligned}
|B| &= \frac{1}{2}|f(t_{n+1}, y(t_{n+1})) - f(t_{n+1}, \hat{y}_{n+1})| \\
&\leq \frac{L}{2}|y(t_{n+1}) - \hat{y}_{n+1}| \\
&\leq \frac{L}{2}\frac{\Delta t^2}{2}|y''(z_1)|
\end{aligned}
$$

where we have assumed and used the Lipschitzianity of f with respect to its second argument, that is, $|f(t, \eta_1) - f(t, \eta_2)| \leq L|\eta_1 - \eta_2|$, and then (1) again. In conclusion

$$
\tau = \max |\tau_n| \leq (1 + L)\frac{\Delta t^2}{4} \max_{t \in [t_0, T]} |y'''(t)| = C\Delta t^2
$$

Thus, Crank-Nicolson scheme is consistent with order of consistency 2.

A closer look at consistency error

We found that:

- for the explicit Euler methods, the consistency error is zero whenever $y'' \equiv 0$, that is, whenever the solution of the Cauchy Problem is a polynomial of degree up to 1.
- the consistency error for Heun method is zero whenever $y''' \equiv 0$, that is, whenever the solution of the Cauchy Problem is a polynomial of degree up to 2.

This suggests that *to have order of consistency p* means that *the scheme computes exactly the solution of the Cauchy Problem whenever this solution is a polynomial of degree up to p*.

This is true in general (no proof provided in this course) and is indeed another easier way of checking consistency of a scheme.

Consistency of implicit methods: implicit Euler

$$\frac{y_{n+1} - y_n}{\Delta t} - f(t_n, y_n) = 0 \quad \forall n$$

Applying this scheme to the exact solution of the Cauchy Problem means considering

$$\tau_n = \frac{y(t_{n+1}) - y(t_n)}{\Delta t} - y'(t_n)$$

which is in general $\neq 0$. Is the order of consistency 1? For this we should have the above $= 0$ when the solution of the Cauchy Problem is $1, t, t^2$.

$$\text{When } y(t) = 1, y' = 0 = f, \Longrightarrow \frac{1 - 1}{\Delta t} - 0 = 0;$$

$$\text{When } y(t) = t, y' = 1 = f, \Longrightarrow \frac{t_{n+1} - t_n}{\Delta t} - 1 = 1 - 1 = 0;$$

Hence, the order of consistency of IE method is 1. Note that

$$\text{When } y(t) = t^2, y' = 2t = f, \Longrightarrow \frac{t_{n+1}^2 - t_n^2}{\Delta t} - 2t_n = t_{n+1} - t_n \neq 0$$

Consistency of implicit methods: Crank-Nicolson

$$\frac{y_{n+1} - y_n}{\Delta t} - \frac{1}{2}\Big(f(t_n, y_n) + f(t_{n+1}, y_{n+1})\Big) = 0 \quad \forall n$$

Applying this scheme to the exact solution means considering

$$\frac{y(t_{n+1}) - y(t_n)}{\Delta t} - \frac{1}{2}\Big(y'(t_n) + y'(t_{n+1})\Big).$$

Let's check the above expression when the solution of the Cauchy Problem is $1, t, t^2$:

$$\text{When } y(t) = 1, y' = 0 = f, \Longrightarrow \frac{1-1}{\Delta t} - \frac{1}{2}(0+0) = 0;$$

$$\text{When } y(t) = t, y' = 1 = f, \Longrightarrow \frac{t_{n+1} - t_n}{\Delta t} - \frac{1}{2}(1+1) = 1 - 1 = 0;$$

$$\text{When } y(t) = t^2, y' = 2t = f, \Longrightarrow \frac{t_{n+1}^2 - t_n^2}{\Delta t} - \frac{1}{2}(2t_n + 2t_{n+1}) = 0$$

Hence, the order of consistency of Crank-Nicolson is 2.

Absolute Stability

The concept of *stability* is another very important and useful concept whose precise definition has to be made precise at various occurrences. Roughly speaking, stability is what guarantees that the errors generated during a numerical procedure do not grow too much.

With Ode stability is a delicate issue, especially when the phenomenon under study has to be followed for a long time. To better see what happens, let us consider a simple model problem, for which we know the exact solution:

$$\begin{cases} y'(t) = ay(t) \quad t > 0 \qquad a \in \mathbb{C} \\ y(0) = y_0 \end{cases}$$

which exact solution is

$$y(t) = y_0 e^{(\mathrm{Re}\, a)t}(\cos((\mathrm{Im}\, a)t) + i \sin((\mathrm{Im}\, a)t))$$

If $\mathrm{Re}(a) > 0$ the exact solution *grows* exponentially. We cannot expect (and we do not want!) that the discrete scheme remains bounded, and it is not even the case to discuss "stability".
Instead, if $\mathrm{Re}(a) < 0$ the exact solution not only is bounded, but *decays* exponentially:

$$a < 0 \longrightarrow |y(t)| \leq |y_0| \quad \textbf{and} \quad \lim_{t\to\infty} |y(t)| = 0.$$

In this case we need to analyse the discrete schemes, and see whether the discrete solution decays too, and behaves like the exact solution. Hence, let $\mathrm{Re}(a) < 0$, and let $\{y_n\}$ be the sequence generated by a numerical scheme. Does $\{y_n\}$ satisfy the following relation?

$$a \in \mathbb{C} \text{ with } \mathrm{Re}\, a < 0 \longrightarrow |y_n| \leq |y_0| \quad \textbf{and} \quad \lim_{n\to\infty} |y_n| = 0?$$

If this happens, the scheme is called *Absolutely stable*, or *A-stable*.

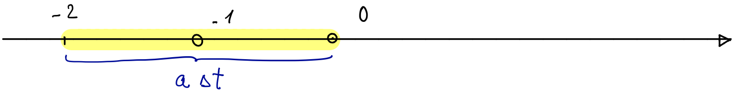
Checking A-stability for Explicit Euler

By applying **Explicit Euler** method to the model problem, we get ($y_{n+1} = y_n + \Delta t f(t_n, y_n)$ with $f(t_n, y_n) = a y_n$)

$$y_{n+1} = (1 + a\Delta t) y_n \quad n = 0, 1, ... \implies y_n = y_0 (1 + a\Delta t)^{n+1}.$$

The exact solution decays exponentially from the initial value y_0, while the growth-decay factor for the discrete scheme is $G = 1 + a\Delta t$.
For having $\lim_{n\to\infty} |y_n| = 0$ we need $|G| < 1$. We observe that $|1 + a\Delta t|$ is the distance between -1 and $a\Delta t$, so $|G| < 1$ if and only if $a\Delta t$ is contained in the unit circle centered in -1.

Stability for Explicit Euler: the real setting

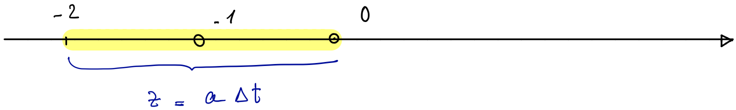


EE is stable $\iff |1 + a\Delta t| < 1$

$\iff -2 < a\Delta t < 0$

$\iff -\frac{2}{a} > \Delta t > 0$ (since $a<0$)

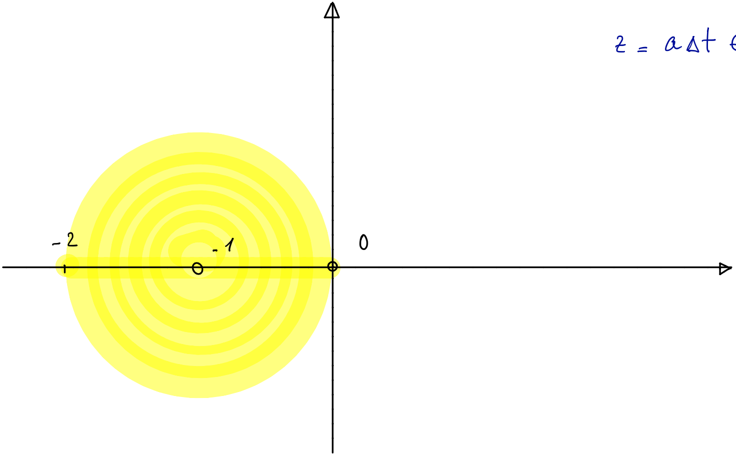
Stability for Explicit Euler: the real setting



$$\text{EE is stable} \iff |1 + a\Delta t| < 1$$

$$\iff -2 < \underbrace{a\Delta t}_{z} < 0$$

Stability for Explicit Euler: the complex setting



$z = a\Delta t \in \mathbb{C}$

EE is stable $\Longleftrightarrow$ $|1 + \overbrace{a\Delta t}^{z}| < 1$

$\Longleftrightarrow$ $z \in$ circle with radius 1 and center -1

It holds:

$$|1 + a\Delta t| < 1 \iff |1 + a\Delta t|^2 < 1$$

but we have

$$|1 + a\Delta t|^2 = (1 + a\Delta t)(1 + \bar{a}\Delta t) = 1 + 2\mathrm{Re}(a)\Delta t + |a|^2\Delta t^2$$

Thus

$$|1 + a\Delta t| < 1 \iff 0 < \Delta t < -\frac{2\mathrm{Re}(a)}{|a|^2} =: \text{A-Stability condition for EE}$$

This is the drawback of Explicit Euler scheme, and of all the explicit schemes: for small enough time steps the stability condition is satisfied, but when $\mathrm{Re}(a)$ is strongly negative (exactly the case of rapid decay in the true solution) we are compelled to keep Δt small.